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Level truncation analysis of exact solutions in open string field theory

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Abstract: We evaluate vacuum energy density of Schnabl's solution using the level truncation calculation and the total action including interaction terms. The level truncated solution provides vacuum energy density expected both for tachyon vacuum and trivial pure gauge. We discuss the role of the phantom term to reproduce correct vacuum energy.

Keywords: String Field Theory, Tachyon Condensation.

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Contents

1. Introduction

String field theory (SFT) provides a non-pertubative framework to analyze various string backgrounds in a unified way. Several classical solutions have been constructed in SFT numerically and analytically, and each of them represents the tachyon vacuum, backgrounds with marginal deformations or rolling tachyon and so on [1]–[18]. The most important progress of recent works in SFT is that Schnabl constructed an analytic classical solution [2] in Witten's open bosonic SFT [19]. The solution is represented as

$$\Psi(\lambda) = \lim_{N\to\infty}\left[\lambda^{N+1}\psi_N - \sum_{n=0}^{N}\lambda^{n+1}\partial_n\psi_n\right],$$

where ψ_n denote wedge states with certain ghost and anti-ghost insertions, and λ is a real parameter. It is believed that for $\lambda = 1$ the solution corresponds to the non-pertubative tachyon vacuum and otherwise the solution should be referred to a trivial pure gauge configuration. It is partly because the above wedge based expression provides correct vacuum energy density expected for the tachyon and the trivial solutions [2].

The crucial difference between the tachyon vacuum and the trivial pure gauge solution seems to be included in the first term of the above expression, so-called the phantom term. Obviously, this term becomes ψ_∞ at $\lambda = 1$, and if $|\lambda| < 1$ it is equal to zero due to the factor λ^{N+1} ($N \to \infty$). Actually, if the first term is not involved in the solution, we can not derive the correct vacuum energy from analytic calculation using quadratic parts of the action [2]. Besides, it is pointed out that the first term is indispensable for the equation of motion contracted with the solution to be satisfied [20, 21]. In other words, the first term is needed to calculate the vacuum energy using the total action with cubic terms, instead of the quadratic action reduced by the equation of motion.

In spite of the important effect of the phantom term, it is known that it becomes to be "zero" also for the case $\lambda = 1$. More precisely, the inner product of ψ_N with any Fock space state vanishes for taking the $N \to \infty$ limit, and therefore the first term is regarded as zero in the Fock space representation. Consequently, it is often said that the phantom term is representative of analytic solutions beyond the Fock space expression.

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Interestingly, the analytic solution is regular from the viewpoint of level truncation as pointed out in the first place [2]. In fact, the solution for $\lambda = 1$ reproduces the correct vacuum energy density in level truncation with respect to the L_0 operator. This energy density was calculated only by using the quadratic action and it is never affected by the phantom term because the truncated solution is a state inside the Fock space. Here, it is natural to ask whether the correct energy density can be reproduced from level truncated calculation using the total action with cubic terms, despite the crucial term is irrelevant. To examine this question is the main motivation in this paper.

In the following, we will calculate the vacuum energy density numerically by truncating the analytic solution and using the action with and without the cubic terms. We will evaluate it for all values of λ providing a regular solution in the level truncation calculation, although only the $\lambda = 1$ case was evaluated so far using the quadratic action. Finally, we will discuss the role of the phantom term to yield the correct vacuum energy density in the last section.

2. Level truncation of the analytic solution

The analytic solution $\Psi(\lambda)$ can be written as

$$\Psi(\lambda) = -\sum_{n=0}^{\infty} \lambda^{n+1} \partial_n \psi_n, \tag{2.1}$$

where we first take the $N \to \infty$ limit and use the fact that $\psi_\infty = 0$. Strictly speaking, this expression is correct in the Fock space representation. From the definition of ψ_n in ref. [2], we can write the solution explicitly as

$$\begin{aligned}\Psi(\lambda) &= -\frac{1}{\pi}\sum_{n=2}^{\infty} \lambda^{n-1} \frac{d}{dn}\left\{U_n^\dagger \left[\frac{n}{\pi}\mathcal{B}_0^\dagger\, \tilde{c}\left(-\frac{\pi}{2}\frac{n-2}{n}\right)\tilde{c}\left(\frac{\pi}{2}\frac{n-2}{n}\right)\right.\right.\\ &\qquad \left.\left. +\tilde{c}\left(-\frac{\pi}{2}\frac{n-2}{n}\right) + \tilde{c}\left(\frac{\pi}{2}\frac{n-2}{n}\right)\right]\right\}|0\rangle\,. \end{aligned}\tag{2.2}$$

This expression is almost the same as that given by Schnabl except for inclusion of λ. After operating the ghost fields on the Fock vacuum, we find

$$(\tilde{c}(-x) + \tilde{c}(x))\,|0\rangle = 2\cos^2 x\, c_1 + 2\sin^2 x\, c_{-1}\,|0\rangle + 2\cos^2 x \tan^4 x\, c_{-3}\,|0\rangle + \cdots \tag{2.3}$$

$$\tilde{c}(-x)\tilde{c}(x)\,|0\rangle = -2\cos^4 x \tan x\, c_0 c_1\,|0\rangle - 2\cos^4 x \tan^3 x (c_0 c_{-1} + c_{-2} c_1)\,|0\rangle \cdots. \tag{2.4}$$

The operator $\mathcal{B}_0^\dagger$ is expanded by negative modes of usual anti-ghost oscillators. The operator $U_n^\dagger$ can be expressed in the canonically ordered form as

$$U_n^\dagger = \cdots e^{u_6\, L_{-6}} e^{u_4\, L_{-4}} e^{u_2\, L_{-2}} \left(\frac{2}{n}\right)^{L_0}, \tag{2.5}$$

where u_n are real numbers as given in ref. [2]. These equations allows us to express the analytic solution as a state in the Fock space.

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For example, let us expand the solution up to level 2:

$$\Psi(\lambda) = t\, c_1 \left|0\right\rangle + u\, c_{-1} \left|0\right\rangle + v\, (\alpha_{-1} \cdot \alpha_{-1}) c_1 \left|0\right\rangle + w\, b_{-2} c_0 c_1 \left|0\right\rangle + \cdots. \tag{2.6}$$

The component fields, t, u, v and w, are given as infinite serieses in the following,

$$t(\lambda) = \sum_{n=2}^{\infty} \lambda^{n-1} \frac{d}{dn} \left[\frac{n}{\pi} \sin^2\left(\frac{\pi}{n}\right) \left(-1 + \frac{n}{2\pi} \sin\left(\frac{2\pi}{n}\right)\right)\right], \tag{2.7}$$

$$u(\lambda) = \sum_{n=2}^{\infty} \lambda^{n-1} \frac{d}{dn} \left[\left(\frac{4}{n\pi} - \frac{n}{\pi} \sin^2\left(\frac{\pi}{n}\right)\right) \left(-1 + \frac{n}{2\pi} \sin\left(\frac{2\pi}{n}\right)\right)\right], \tag{2.8}$$

$$v(\lambda) = \sum_{n=2}^{\infty} \lambda^{n-1} \frac{d}{dn} \left[\left(\frac{2}{3n\pi} - \frac{n}{6\pi}\right) \sin^2\left(\frac{\pi}{n}\right) \left(-1 + \frac{n}{2\pi} \sin\left(\frac{2\pi}{n}\right)\right)\right], \tag{2.9}$$

$$w(\lambda) = \sum_{n=2}^{\infty} \lambda^{n-1} \frac{d}{dn} \left[\sin^2\left(\frac{\pi}{n}\right) \left(\frac{8}{3n\pi} - \frac{2n}{3\pi} + \frac{n^2}{3\pi^2} \sin\left(\frac{2\pi}{n}\right)\right)\right]. \tag{2.10}$$

All of these converge absolutely if $|\lambda| \leq 1$ and the same is true up level 10. A component field is given by a power series as $\sum_{n=2}^{\infty} \lambda^{n-1} a_n$ if it is non-zero. We can easily check that the radius of convergence is 1 for these component fields up to level 10. For $|\lambda| = 1$, we expand as $a_n/a_{n+1} = 1 + h/n + \mathrm{O}(1/n^2)$ and we can find $h > 1$ up to level 10. Hence, the series is convergent for $|\lambda| \leq 1$.

The expression (2.1) satisfies the equation of motion for arbitrary λ, that is proved only by using the identity of ψ_n irrelevant to λ [2]. It is not clear for what range of λ the solution should be defined. However, λ must take the value between -1 and 1 if the solution has a well-defined Fock space expression.

It is difficult to derive analytic expressions for these serieses. If we expand coefficients in the series in powers of $1/n$, only terms $1/n^4$, $1/n^6$, $1/n^8$, $\cdots$ appear in it. Therefore, we can sum up the series numerically with extreme precision as mentioned in ref. [2]. These fields obey a symmetry generated by $K_1^{\rm matter}$ for any λ as in case of $\lambda = 1$ [2]. Using this symmetry, we can check numerically whether the calculated result is correct or not. The resulting plots of the above fields are depicted in figure 1. These values at $\lambda = 1$ coincides with those of earlier results in ref. [2]. Each curve for component fields smoothly varies from zero at $\lambda = 0$. It has no discontinuity even at $\lambda = 1$, despite the vacuum energy should fall down from zero to the minus energy at $\lambda = 1$.

While the $\lambda \neq 1$ case is expected to be trivial pure gauge, the $\lambda = -1$ case is exceptional because the solution for $\lambda = -1$ satisfies the symmetry

$$(-1)^{\mathcal{L}_0} \mathcal{L}_0 \Psi = \mathcal{L}_0 \Psi.$$

It is the same symmetry satisfied by the tachyon vacuum solution and therefore the solution in that case may be regarded as a non-trivial vacuum [2]. However, at the $\lambda = -1$, all component fields are continuous similar to those of $\lambda = 1$.

Now, let us compute the vacuum energy using the action with the quadratic terms only. Using the equation of motion, the vacuum energy density is given by

$$V_q(\lambda) = \frac{\pi^2}{3} \left\langle \Psi(\lambda),\, Q_B \Psi(\lambda) \right\rangle, \tag{2.11}$$

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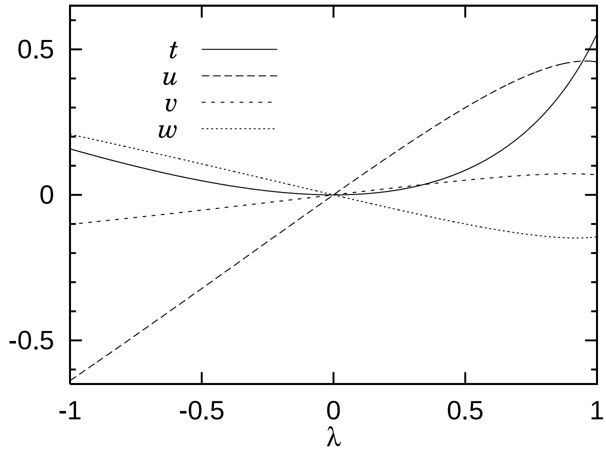

Figure 1: Component fields up to level 2. Each curve consists of two thousand plots with lines.

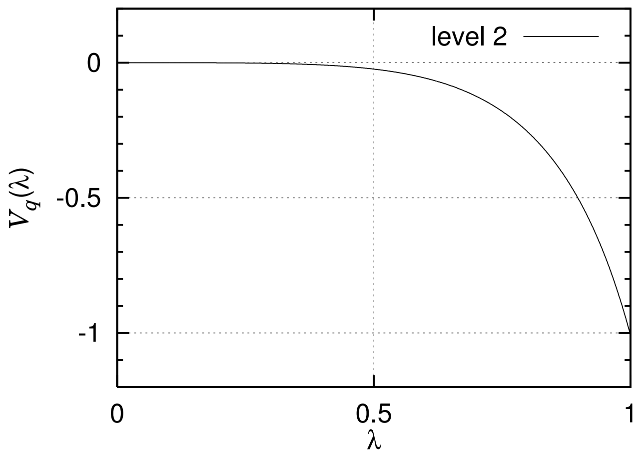

Figure 2: The energy density for level 2. It is calculated only with the quadratic terms of the action.

where it is normalized as to be minus one for the tachyon vacuum. Substituting the truncated solution into it, we can calculate the energy density as a function of λ. For level 2 and $0 \leq \lambda \leq 1$, we make a graph of the vacuum energy density in figure 2. As given in ref. [2], the energy density at $\lambda = 1$ is good agreement with the correct density even at level 2. For $0 < \lambda \lesssim 0.3$, the energy density is almost zero and it is well-behaved as a pure gauge solution. We can not understand this zero energy density trivially by the values of component fields in figure 1. This good property can be regarded as a result of cancellation of each contribution of all component fields.

We consider higher level approximation for full range of λ. We have computed component fields up to level 10. We display the result in figure 3. Around $\lambda = 0$, we have

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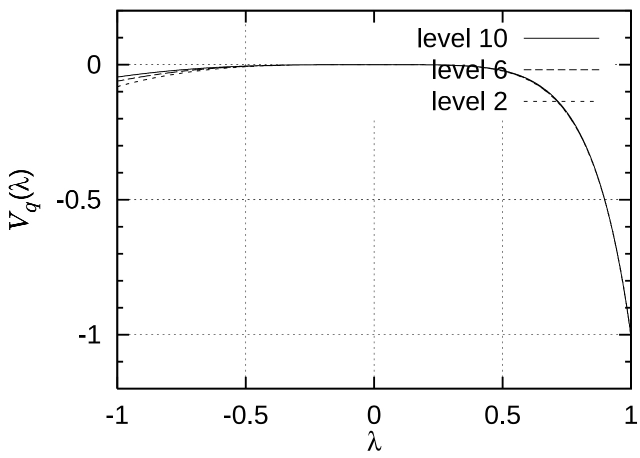

Figure 3: The energy densities up to level 2, 6, 10 which are calculated only with the quadratic terms of the action. Each line is drawn as two thousand points with lines.

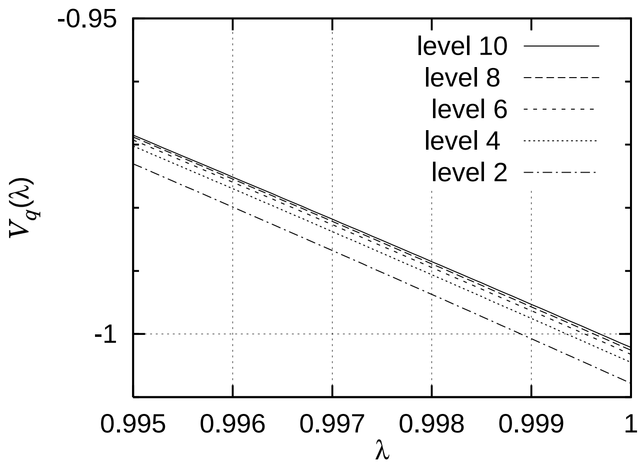

Figure 4: The energy densities up to level 10 which are calculated only with the quadratic terms of the action. Each line is drawn as one thousand points with lines.

almost zero energy density for all level. For $\lambda \sim -1$, the energy density approaches zero as the truncation level is increased. This result is consistent with the expectation that the solution is a trivial pure gauge solution for $-1 \leq \lambda < 1$. Around nearby $\lambda = 1$, we can not distinguish each curve from the others. So, we enlarge the resulting plots for higher levels around $\lambda \sim 1$ in figure 4. We find that the energy density approaches slowly but gradually to the correct value as the approximation level is increased. Consequently, although the plots may approach a critical curve for higher levels, the result is consistent with the expectation that, if the truncation level goes to infinity, the plots approach to the step

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	$L=0$	$L=2$	$L=4$	$L=6$	$L=8$	$L=10$
$(L,\,2L)$	-0.577920	-1.081077	-1.054081	-1.036779	-1.025645	-1.018552
$(L,\,3L)$	-0.577920	-1.065177	-1.047979	-1.032868	-1.023261	—
quad. terms	-1.007766	-1.007815	-1.004499	-1.003217	-1.002556	-1.002130

Table 1: Energy density calculated by the full action. For comparison, energy density calculated by the quadratic action is listed in the last row.

function,

$$f(\lambda)=\begin{cases}0 & (-1\le\lambda<1)\\ -1 & (\lambda=1).\end{cases} \tag{2.12}$$

Next, let us compute the energy density using the total action including cubic terms, which is given by

$$V_f(\lambda)=\frac{\pi^2}{2}\left\langle\Psi(\lambda),\,Q_B\Psi(\lambda)\right\rangle+\frac{\pi^2}{3}\left\langle\Psi(\lambda),\,\Psi(\lambda)*\Psi(\lambda)\right\rangle, \tag{2.13}$$

where we have used the same normalization as before.

For $\lambda=1$, the resulting energy density is summarized in the following table. For level zero, the energy density is $-0.57\cdots$ and it is about one-half of the correct density. But, it is comparable to the level zero result in Siegel gauge, $-0.68\cdots$. The both results of $(L,2L)$ and $(L,3L)$ are almost the same. The level 6 and 8 results of $(L,3L)$ are closer to -1 than that of $(L,2L)$. Up to level 10, the energy density agrees with the correct value to 10^{-1}, but that is worse than the result calculated by the quadratic action. However, we can find that the resulting energy approaches to the expected value -1 as the truncation level is increased.

We proceed to consider the full range of λ. In figure 5 and 6, we display the energy density evaluated by the truncated actions of $(L,2L)$ and $(L,3L)$, respectively. Around nearby $\lambda=0$, the energy density is almost equal to zero. Around $\lambda=-1$, the energy density approaches to zero as the truncation level is increased, but from the positive energy region as contrasted to the calculation by the quadratic action. Even in the case using the total action, we can find that the resulting plots gradually become the step function as the level is increased.

3. Discussions

We calculated the vacuum energy density for the analytic classical solution constructed by Schnabl using its Fock space expression. We found that, as the truncation level is increased, the resulting plots approach to the step function for $|\lambda|\le 1$. The result is consistent with the fact that the solution for $\lambda=1$ is the tachyon vacuum solution and otherwise it corresponds to a trivial pure gauge solution. In particular, our calculation suggests that the solution for $\lambda=-1$ is also trivial, although it possesses the same symmetry as the

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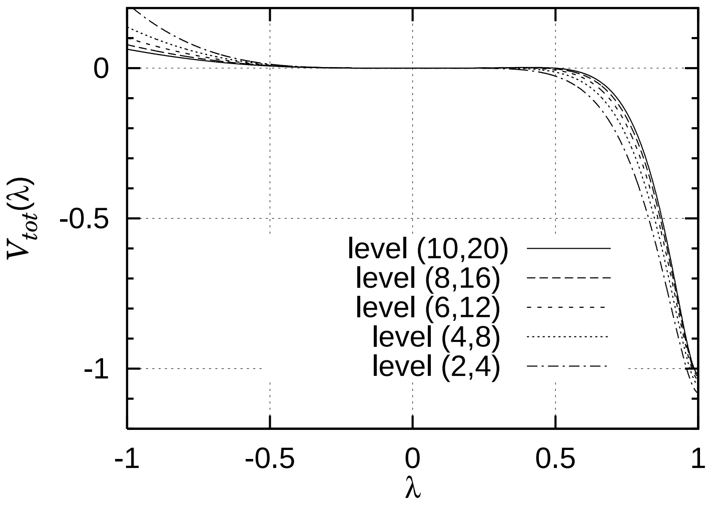

Figure 5: Energy density evaluated by the $(L, 2L)$ truncated action.

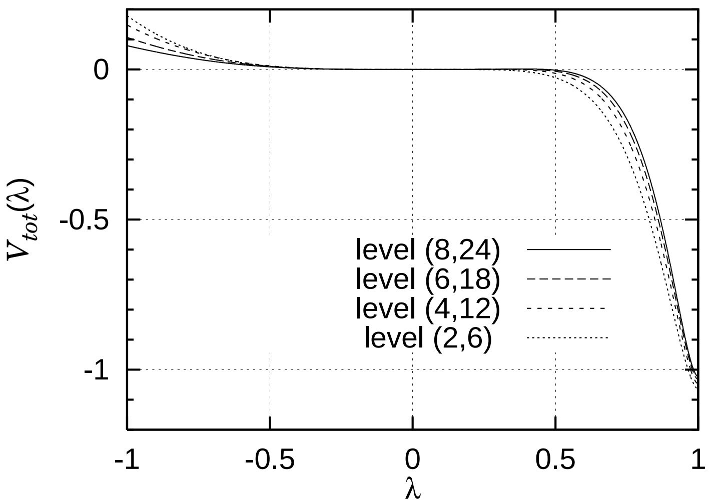

Figure 6: Energy density evaluated by the $(L, 3L)$ truncated action.

tachyon vacuum solution. Consequently, the analytic solution is well-behaved for $|\lambda| \leq 1$ from the point of view of level truncation.

Our analysis was based on level truncation calculation and therefore the phantom term does not contribute to the vacuum energy density. Our result suggests that the phantom term is not indispensable to reproduce the correct vacuum energy, although the phantom term is an important ingredient to evaluate the vacuum energy analytically.

It is no wonder that the phantom term plays a whole different role in each expression of the solution. Because, a string field is given as a state in the Hilbert space with an indefinite

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metric, namely string field theory does not possess a positive definite norm. Actually, the correct vacuum energy was reproduced as a result of cancellation between positive and negative infinite energy. This fact can be found most clearly in the solution expanded in $\mathcal{L}_0$ eigen-states. Using the solution truncated with respect to the $\mathcal{L}_0$ level, we find that the vacuum energy density is not convergent as the truncation level is increased. But, surprisingly, the Padé approximation to the divergent series can reproduce correct vacuum energy both for $\lambda = 1$ [2] and $\lambda \neq 1$ [22]. This result of the $\mathcal{L}_0$ truncation indicates that the vacuum energy density for the solution is given as a conditionally convergent series.

Hence, to define the analytic solution, an important point is how to regularize it in SFT based on the indefinite metric. In the wedge based expression, the integer N seems to be a kind of regularization parameter. The phantom term is important only if we regularize the solution in terms of N. Our results suggest that the truncation level L can be regarded as a good regularization parameter as well as N. If that is the case, we will find that the vacuum energy from the truncated solution agrees with the correct value as the L goes to infinity, that is

$$\lim_{\epsilon \to 0} \frac{\pi^2}{3} \left\langle e^{-\epsilon L_0}\Psi(\lambda),\, Q_B e^{-\epsilon L_0}\Psi(\lambda)\right\rangle = \begin{cases} -1 & (\lambda = 1) \\ 0 & (-1 \leq \lambda < 1). \end{cases} \tag{3.1}$$

We expect a similar behavior for the vacuum energy including the cubic terms. In any case, the phantom term has no effect on the conjectural equation.

The parameters N and L seem to provide a sort of "correct" regularization. Eventually, a crucial issue is how we can regularize the solution or the theory "correctly". It is well-known that symmetry is a key role to regularize a quantum field theory of gauge fields, which is formulated in the framework of the indefinite-metric theory. In contrast, we still lack the criterion for "correct" regularization in string field theory. For example, the vacuum energy for the identity-based solution is given as an indefinite quantity [18, 23]. We hope that the Schnabl's solution will help to find a good way to regularize string field theory in order to search non-pertubative vacua further.

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